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Art XIII.—*Singular Parameter Values in the Boundary Problems of the Potential Theory.*

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The method of integral equations has been successfully applied to the boundary problems requiring the determination of potentials, $W(p)$ and $V(p)$ satisfying the boundary relations[1] —

$$(1) \qquad \begin{cases} \frac{1}{2}[W(t^+) - W(t^-)] - \frac{1}{2}\lambda[W(t^+) + W(t^-)] = f(t) \\ \frac{1}{2}\left[\frac{dV}{dn}(t^-) - \frac{dV}{dn}(t^+)\right] - \frac{1}{2}\lambda\left[\frac{dV}{dn}(t^-) + \frac{dV}{dn}(t^+)\right] = f(t) \end{cases}$$

respectively, whether the potentials are ordinary[2] corresponding to Laplace's equation, or "generalised" corresponding to the equation—

$$(2) \qquad \nabla^2 U - k^2 U = 0$$

The latter potential I have considered in a paper[3] recently communicated to the Quarterly Journal. It is shewn that solutions to the problems can be uniquely determined, except for certain singular values of the parameter, in the form of potentials of double and simple strata respectively, given by[4]—

$$(3) \qquad \begin{cases} W(p) = \int f(t) H(tp) dt \\ V(p) = \int G(pt) f(t) dt. \end{cases}$$

At a singular parameter value λ_0, however, the solutions become infinite, since each of the functions $H(tp)$ and $G(pt)$ has a simple pole, unless certain conditions are satisfied. It will be shewn that the parts $\mathcal{H}(tp)$ and $\mathcal{G}(pt)$ of these functions remaining finite at the pole λ_0, form the corresponding functions for the solutions at this pole of the problems (1), which, however, must be modified[5] in their second members. The residues $P(tp)$ and $Q(pt)$ of $H(tp)$ and $G(pt)$ respectively, also play an important part in the following argument.

1 Poincaré. "Sur les équations de la Physique." Rendiconti, Palermo, 1894.

2 Plemelj. Monatsfefte für Math. und Physik, Bd. 15, S. 337-411 (1904); Bd. 18, S. 180-211 (1907).

3 "Boundary problems for the generalised potential corresponding to the equation $\nabla^2 U - k^2 U = 0$." Quarterly Journal, vol. 46, pp. 66-82.

4 The integration throughout is extended over the boundary of the region considered, unless otherwise stated. The notation of my previous paper is adhered to.

5 Weatherburn, loc. cit. § 6; also Plemelj, loc. cit., S. 404-5.

Plemelj's work[1] is confined to the ordinary potential and deals chiefly with the pole $\lambda = +1$. The present paper extends the investigation to the generalised potential, and also to the general pole λ_0. For this characteristic number, which may be any whatever, more general relations are established connecting the residues and the functions $H(tp)$ and $G(pq)$, which correspond to the modified problems. The boundary discontinuities of these functions and their derivatives are investigated, and also certain theorems of reciprocity. Expansions for the various functions are found as power series in the parameter λ.

In the first part of the paper the investigation applies to the ordinary and generalised potentials alike. In the second part the ordinary potential is considered separately, and results peculiar to Laplace's equation are obtained which depend either upon the fact that $\lambda = \pm 1$ are here characteristic numbers, or upon the special value of the integral of $h(tp)$ extended over the boundary. Values for the boundary integrals of the different functions are investigated. Further from the convergence of the above expansions when $|\lambda| = 1$ a value is deduced for the conductor potential. It will also be shown that the solutions of the second boundary problem for the inner and outer regions are expressible in terms of a single function.

Finally the case of the generalised potential is considered separately. The value is found of the integral of $h(tp)$ extended over the boundary, in terms of the potential of a space distribution of matter. Further relations are found connecting the boundary integrals of the other functions involved.

I.—*Ordinary and generalised potentials.*

§1. *Solutions and their poles.* The solutions of the boundary problems as given by (3), when expressed in terms of the resolvent H(ts) become[2]—

$$(3') \qquad \begin{cases} \mathrm{W}(p) = \int \mathrm{f}(t)[h(tp) + \lambda \int \mathrm{H}(t\theta) h(\theta p) d\theta] dt \\ \mathrm{V}(p) = \int [g(pt) + \lambda \int g(p\theta) \mathrm{H}(\theta t) d\theta] \mathrm{f}(t) dt \end{cases}$$

where

$$h(\theta p) = \frac{d}{dn} g(\theta p)$$

θ being a point on the boundary, and $g\ (pq)$ is a particular solution of Laplace's equation if the potential is ordinary, and of the equation (2) if it is generalised. The value of this function is given by—

1 Cf. also "Potentialtheoretische Untersuchungen," Teubner, Leipzig (1911).

2 Cf. Weatherburn. Loc. cit. § 2.

(4) $$\begin{cases} g(pq) = \frac{1}{\pi}\log\frac{1}{r} \text{ for the logarithmic potential} \\ g(pq) = \frac{1}{2\pi}\cdot\frac{1}{r} \text{ for the Newtonian potential} \end{cases}$$

which are solutions of Laplace's equation; and

(4′) $$\begin{cases} g(pq) = \frac{1}{\pi}f(kr) \text{ for the plane} \\ g(pq) = \frac{1}{2\pi}\cdot e^{-kr}/r \text{ for space} \end{cases}$$

when the potential is generalised corresponding to the equation (2). In this r is the radius vector joining the points p and q, and $f(z)$ has the same meaning as in my paper already referred to. The functions $H(tp)$ and $G(pt)$ are equal to the corresponding expressions of (3′) in square brackets. The former is an extension of the solving function in which any point p replaces the boundary point s. The latter may be defined more generally for any two points pq by—

$$G(pq) = g(pq) + \lambda\int g(p\theta)H(\theta q)d\theta$$

This function is the Green's function[1] for the boundary problems (1). It will be seen that $H(tp)$ can be expressed in terms of it by normal differentiation, so that both solutions (3) can be given in terms of it by a representation of Green's type. It is easily verified that

$$\int g(qt)H(tp)dt = \int G(qt)h(tp)dt$$

so that the equations defining and connecting these functions are—

(5) $$\begin{cases} H(tp) - h(tp) = \lambda\int h(t\theta)H(\theta p)d\theta = \lambda\int H(t\theta)h(\theta p)d\theta \\ G(qp) - g(qp) = \lambda\int g(q\theta)H(\theta p)d\theta = \lambda\int G(q\theta)h(\theta p)d\theta \end{cases}$$

Now when λ is equal to a characteristic number (singular value) λ_0, each of the functions $H(tp)$ and $G(qp)$ has a simple pole.[2] The solutions expressed by (3) are therefore infinite, and cease to have a meaning. Since the pole is simple we may write—

(6) $$\begin{cases} H(tp) = H(tp) + \dfrac{P(tp)}{\lambda_0 - \lambda} \\ G(qp) = G(qp) + \dfrac{\lambda_0 Q(qp)}{\lambda_0 - \lambda} \end{cases}$$

where $H(tp)$ and $G(qp)$ are functions of λ, which depend on λ_0 and remain finite when $\lambda = \lambda_0$; the residues $P(tp)$ and $\lambda_0\, Q(qp)$ do not involve λ but depend on λ_0. If now we substitute from (6) in (5), multiply by $(\lambda_0 - \lambda)$ and proceed to the limit $\lambda = \lambda_0$, we obtain the following relations:—

(7) $$\begin{cases} P(tp) = \lambda_0\int P(t\theta)h(\theta p)d\theta = \lambda_0\int h(t\theta)P(\theta p)d\theta \\ Q(qp) = \int g(q\theta)P(\theta p)d\theta = \lambda_0\int Q(q\theta)h(\theta p)d\theta \end{cases}$$

1 Cf. Weatherburn. "Green's Functions for the equation $\Delta^2 u - k^2 u = 0$, etc." Quarterly Journal, vol. 46. The remaining references are to my earlier paper.

2 Weatherburn. Loc. cit. § 3.

If again we substitute from (6) in (5) and use the relations (7) we find—

$$(8)\quad \begin{cases} \lambda\int h(t\theta)H(\theta p)d\theta = \lambda\int H(t\theta)h(\theta p)d\theta = H(tp) - h(tp) + \frac{1}{\lambda_0}\cdot \mathrm{P}(tp) \\ \lambda\int g(q\theta)H(\theta p)d\,\theta = \lambda\int G(q\theta)h(\theta p)d\,\theta = G(qp) - g(qp) + \mathrm{Q}(qp) \end{cases}$$

These relations are more general than those found for the ordinary potential by Plemelj, who considers mainly the pole $\lambda = +1$. They play an important part in our argument.

The value of $\mathrm{P}(ts)$ is known, being the residue of the resolvent for the simple pole λ_0. If m be the order of multiplicity of the root λ_0 of the determinant $\mathrm{D}(\lambda)$, $\mathrm{P}(ts)$ may be expressed as the sum

$$(9)\qquad \mathrm{P}(ts) = \phi_1(t)\psi_1(s) + \phi_2(t)\psi_2(s) + \ldots + \phi_m(t)\psi_m(s)$$

where the functions ϕ_i, $\psi_i (i = 1, 2, \ldots m)$ are the m linearly independent solutions of the homogeneous integral equations.

$$\begin{cases} \phi(t) = \lambda_0\int h(t\theta)\phi(\theta)d\theta \\ \psi(t) = \lambda_0\int \psi(\theta)h(\theta t)d\theta \end{cases}$$

satisfying the usual orthogonal relations. Hence the values of $\mathrm{P}(tp)$ and $\mathrm{Q}(qp)$ are given by

$$(9')\qquad \begin{cases} \mathrm{P}(tp) = \phi_1(t)\psi_1(p) + \ldots\ldots + \phi_m(t)\psi_m(p) \\ \mathrm{Q}(qp) = \Phi_1(q)\psi_1(p) + \ldots\ldots + \Phi_m(q)\psi_m(p) \end{cases}$$

where $\Phi(q)$ is the potential of a simple stratum of density $\phi(t)$ over the boundary, and $\psi(p)$ is that of a double stratum of moment $\lambda_0\psi(t)$.

If we introduce the functions

$$(10)\qquad \begin{cases} k(tp) = h(tp) - \frac{1}{\lambda_0}\mathrm{P}(tp) \\ l(qp) = g(qp) - \mathrm{Q}(qp) \end{cases}$$

we are enabled to express (8) in a form exactly similar to (5). For if in the first of (8) we replace p by θ, multiply throughout by $\mathrm{P}(\theta p)$ and integrate over the boundary, we find in virtue of (7) that

$$\int H(t\theta)\mathrm{P}(\theta p)d\theta = \int \mathrm{P}(t\theta)H(\theta p)d\theta = 0.$$

Similarly it may be proved that

$$\int G(q\theta)\mathrm{P}(\theta p)d\theta = \int \mathrm{Q}(q\theta)H(\theta p)d\theta = 0.$$

These integrals may therefore be combined with the integrals in (8) without altering their values, so that the relations may be written

$$(11)\qquad \begin{cases} \lambda\int k(t\theta)H(\theta p)d\theta = \lambda\int H(t\theta)k(\theta p)d\theta = H(tp) - k(tp) \\ \lambda\int l(q\theta)H(\theta p)d\theta = \lambda\int G(q\theta)k(\theta p)d\theta = G(qp) - l(qp) \end{cases}$$

which are of the same form as (5); but $G(qp)$, as will be seen, is the Green's function for the modified problems, and $H\ (tp)$ bears the same relation to it that $\mathrm{H}(tp)$ bears to $\mathrm{G}(qp)$.

§2.—*Boundary discontinuities.*—The second of equations (9) shows that Q (qp), regarded as a function of q, is the potential of a simple stratum of density $P(\theta p)$. From the boundary properties of such it follows that

$$\begin{cases} \frac{1}{2}\left[\frac{d}{dn}Q(t^-p) - \frac{d}{dn}Q(t^+p)\right] = P(tp) \\ \frac{1}{2}\left[\frac{d}{dn}Q(t^-p) + \frac{d}{dn}Q(t^+p)\right] = \int h(t\theta)P(\theta p)d\theta = P(tp)/\lambda_0 \end{cases}$$

Adding and subtracting we find for the normal derivative of $Q(qp)$ on either side of the boundary

$$(12) \qquad \begin{cases} \lambda_0\frac{d}{dn}Q(t^-p) = (1+\lambda_0)P(tp) \\ \lambda_0\frac{d}{dn}Q(t^+p) = (1-\lambda_0)P(tp) \end{cases}$$

Regarded, however, as a function of p, Q (qp) is a double stratum potential of moment $\lambda_0 Q(q\theta)$. Hence

$$\begin{cases} \frac{1}{2}[Q(qt^+) - Q(qt^-)] = \lambda_0 Q(qt) \\ \frac{1}{2}[Q(qt^-) + Q(qt^+)] = \lambda_0\int Q(q\theta)h(\theta t)d\theta = Q(qt) \end{cases}$$

Adding and subtracting we have for the values of Q (qp) on either side of the boundary

$$(13) \qquad \begin{cases} Q(qt^+) = (1+\lambda_0)Q(qt) \\ Q(qt^-) = (1-\lambda_0)Q(qt) \end{cases}$$

Similarly P (sp) as a function of p is a double stratum of moment $\lambda_0 P(s\theta)$; and its values on either side of the boundary are found in the same way to be

$$(14) \qquad \begin{cases} P(st^+) = (1+\lambda_0)P(st) \\ P(st^-) = (1-\lambda_0)P(st) \end{cases}$$

From the second of equations (10) $G(qp)$, regarded as a function of q is the sum of potentials $g(qp), -Q(qp)$, and a simple stratum of density $\lambda H(\theta p)$. From the behaviour of these at the boundary, and in virtue of (12), it follows

$$\frac{1}{2}\left[\frac{d}{dn}G(t^-p) - \frac{d}{dn}G(t^+p)\right] = \lambda H(tp) - P(tp)$$

$$\frac{1}{2}\left[\frac{d}{dn}G(t^-p) + \frac{d}{dn}G(t^+p)\right] = \lambda\int h(t\theta)H(\theta p)d\theta + h(tp) - P(tp)/\lambda_0$$
$$= H(tp)$$

Adding and substracting we find

$$(15) \qquad \begin{cases} \frac{d}{dn}G(t^-p) = (1+\lambda)H(tp) - P(tp) \\ \frac{d}{dn}G(t^+p) = (1-\lambda)H(tp) + P(tp) \end{cases}$$

Regarded, however, as a function of p, $G(qp)$ is a double stratum potential of moment $\lambda G(q\theta)$, together with potentials $g(qp)$ and $-Q(qp)$. From the boundary properties of these we deduce

$$(16)\qquad \begin{cases} G(qt^+) = (1+\lambda)G(qt) - \lambda_0 Q(qt) \\ G(qt^-) = (1-\lambda)G(qt) + \lambda_0 Q(qt) \end{cases}$$

Finally $H(sp)$ regarded as a function of p is the sum of potentials $h(sp)$, $-P(sp)/\lambda_0$, and a double stratum of moment $\lambda H(s\theta)$. From which it follows, in virtue of (14) that

$$(17)\qquad \begin{cases} H(st^+) = (1+\lambda)H(st) - P(st) \\ H(st^-) = (1-\lambda)H(st) + P(st) \end{cases}$$

§3.—*Solution regular at a singular parameter value.*—We are now in a position to find solutions to the boundary problems (1), with second members modified, having no singularities for the characteristic number λ_0. If we define the functions $W(p)$ and $V(p)$ by

$$(18)\qquad \begin{cases} W(p) = \int f(\theta) H(\theta p) d\theta \\ V(p) = \int G(p\theta) f(\theta) d\theta \end{cases}$$

we find on substituting the values of $H(\theta p)$ from (8) that $W(p)$ is the sum of potentials of double strata of moments $f(t)$, $-\int f(\theta) P\theta t) d\theta$, and $\lambda \int f(\theta) H(\theta t) d\theta$ respectively.

Hence we find that

$$\begin{aligned} &\tfrac{1}{2}[W(t^+) - W(t^-)] - \tfrac{1}{2}\lambda[W(t^+) + W(t^-)] \\ &= f(t) - \int f(\theta) P(\theta t) d\theta + \lambda \int f(\theta) H(\theta t) d\theta \\ &\quad - \lambda \int \{ f(\phi) h(\phi t) - \int f(\theta) P(\theta\phi) h(\phi t) d\theta + \lambda \int f(\theta) H(\theta\phi) h(\phi t) d\theta \} d\phi \end{aligned}$$

In virtue of (8) the second member disappears except for the first two terms. So that $W(p)$ satisfies the boundary condition.

$$(19a)\qquad \tfrac{1}{2}[W(t^+) - W(t^-)] - \tfrac{1}{2}\lambda[W(t^+) + W(t^-)] = f(t) - \int f(\theta) P(\theta t) d\theta$$

In this all the functions are regular when $\lambda = \lambda_0$; so that this equation admits the solution $W(p)$ which is regular even when λ is put equal to the singular value λ_0. It has been shown elsewhere[1] that for this value of the parameter the first problem (1) does not admit a solution by double stratum unless the condition

$$\int f(\theta) P(\theta t) d\theta = 0$$

is satisfied, in which case the solution is obviously $W(p)$.

Similarly substituting the value of $G(p\theta)$ given by (8) we find that $V(p)$ is the sum of potentials of simple strata of densities $f(t)$, $-\int P(t\theta) f(\theta) d\theta$ and $\lambda \int H(t\theta) f(\theta) d\theta$. From the boundary properties of simple strata it follows that

$$\begin{aligned} &\tfrac{1}{2}\left[\frac{d}{dn} V(t^-) - \frac{d}{dn} V(t^+)\right] - \tfrac{1}{2}\lambda\left[\frac{d}{dn} V(t^-) + \frac{d}{dn} V(t^+)\right] \\ &= f(t) - \int P(t\theta) f(\theta) d\theta + \lambda \int H(t\theta) f(\theta) d\theta \\ &\quad - \lambda \int h(t\phi) \left[f(\phi) - \int P(\phi\theta) f(\theta) d\theta + \lambda \int H(\phi\theta) f(\theta) d\theta \right] d\phi \end{aligned}$$

1 Weatherburn. "Boundary Problems, etc," §6.

In virtue of (7) and (8) the second member reduces to the first two terms; so that $V(p)$ satisfies the boundary problem.

$$(19b)\quad \tfrac{1}{2}\Big[\frac{d}{dn}V(t^-)-\frac{d}{dn}V(t^+)\Big]-\tfrac{1}{2}\lambda\Big[\frac{d}{dn}V(t^-)+\frac{d}{dn}V(t^+)\Big]$$
$$=\mathrm{f}(t)-\int \mathrm{P}(t\theta)\mathrm{f}(\theta)d\theta$$

All the functions involved are regular for the singular value $\lambda=\lambda_0$, so that $V(p)$ is the solution of the problem (19b) regular even when λ is equal to this singular value. The problem (1b) does not admit a solution by simple stratum only, when $\lambda=\lambda_0$, unless the condition

$$\int \mathrm{P}(t\theta)\mathrm{f}(\theta)d\theta=0$$

is satisfied, in which case the required solution is obviously $V(p)$. The problems (19), derived from (1) by altering the second member, we shall speak of as the modified problem for the singular value λ_0. The functions $H(tp)$ and $G(pt)$ bear the same relation to the solution of the modified problems that $\mathrm{H}(tp)$ and $\mathrm{G}(pt)$ bear to the original problems (1).

§4.—*Expansions.*—From the formulæ (8) and (18) we may obtain, by the method of successive approximations, expansions for the various functions in ascending powers of λ. These are certainly true for $|\lambda|<1$, and in particular cases even for $|\lambda|=1$. For the present we shall assume that the absolute value of λ is less than unity.

Thus from (8) in virute of (7) we find

$$(20)\quad \begin{cases} H(ts)=\Big[h(ts)-\frac{1}{\lambda_0}\mathrm{P}(ts)\Big]+\lambda\Big[h_1(ts)-\frac{1}{\lambda_0{}^2}\mathrm{P}(ts)\Big] \\ \qquad\qquad +\lambda^2\Big[h_2(ts)-\frac{1}{\lambda_0{}^3}\mathrm{P}(ts)\Big]+\dots \\ G(ps)=[g(ps)-\mathrm{Q}(ps)]+\lambda\Big[g_1(ps)-\frac{1}{\lambda_0}\mathrm{Q}(ps)\Big] \\ \qquad\qquad +\lambda^2\Big[g_2(ps)-\frac{1}{\lambda_0{}^2}\mathrm{Q}(ps\Big]+\dots \end{cases}$$

where the suffixes denote functions formed by successive operations as

$$h_1(ts)=\int h(t\theta)h(\theta s)d\theta,$$
$$h_2(ts)=\int h_1(t\theta)h(\theta s)d\theta, \text{ etc.}$$

and

$$g_1(ps)=\int g(p\theta)h(\theta s)d\theta,$$
$$g_2(ps)=\int g_1(p\theta)h(\theta s)d\theta, \text{ etc.}$$

If we extend the notation and replace s by any point p we may write

$$h_1(tp)=\int h(t\theta)h(\theta p)d\theta,$$
$$h_n(tp)=\int h_{n-1}(t\theta)h(\theta p)d\theta, \text{ etc.}$$

and the first equation (20) becomes

$$(20') \quad \Big\{ H(tp)=\Big[h(tp)-\frac{1}{\lambda_0}\mathrm{P}(tp)\Big]+\lambda\Big[h_1(tp)-\frac{1}{\lambda_0^2}\mathrm{P}(tp)\Big]+\ldots$$

Introducing these values in (18) we have, for the solutions of the boundary problems (19)

$$(21) \quad \begin{cases} W(p)=\int \mathrm{f}(\theta)\Big\{\Big[h(\theta p)-\frac{1}{\lambda_0}\mathrm{P}(\theta p)\Big]+\lambda\Big[h_1(\theta \mathrm{p})-\frac{1}{\lambda_0^2}\mathrm{P}(p\theta)\Big]+\ldots\Big\}d\theta. \\ V(p)=\int\Big\{[g(p\theta)-\mathrm{Q}(p\theta)]+\lambda\Big[g_1(p\theta)-\frac{1}{\lambda_0}\mathrm{Q}(p\theta)\Big]+\ldots\Big\}\mathrm{f}(\theta)d\theta. \end{cases}$$

We may further obtain expansions for the moment $\nu(t)$, and the density $\mu(t)$ of the strata satisfying (19); for these are solutions of the integral equations

$$\begin{cases} \nu(t)-\lambda\int\nu(\theta)h(\theta t)d\theta=\mathrm{f}(t)-\int\mathrm{f}(\theta)\mathrm{P}(\theta t)d\theta=\mathrm{E}(t), \text{ say} \\ \mu(t)-\lambda\int h(t\theta)\mu(\theta)d\theta=\mathrm{f}(t)-\int\mathrm{P}(t\theta)\,\mathrm{f}(\theta)d\theta=\mathrm{F}(t), \text{ say,} \end{cases}$$

and are therefore given by the expansions

$$(22) \quad \begin{cases} \nu(t)=\mathrm{E}(t)+\lambda\mathrm{E}_1(t)+\lambda^2\mathrm{E}_2(t)+\ldots\ldots \\ \mu(t)=\mathrm{F}(t)+\lambda\mathrm{F}_1'(t)+\lambda_2\mathrm{F}_2'(t)+\ldots\ldots \end{cases}$$

where the successive functions are given by

$$\mathrm{E}_1(t)=\int\mathrm{E}(\theta)h(\theta t)d\theta$$
$$\mathrm{E}_2(t)=\int\mathrm{E}_1(\theta)h(\theta t)d\theta, \text{ \&c.}$$

and

$$\mathrm{F}_1'(t)=\int h(t\theta)\mathrm{F}(\theta)d\theta$$
$$\mathrm{F}_2'(t)=\int h(t\theta)\mathrm{F}_1'(\theta)d\theta, \text{ \&c.}$$

If we evaluate these functions we find

$$\mathrm{E}_n(t)=\int\mathrm{f}(\theta)h_{n-1}(\theta t)d\theta-\frac{1}{\lambda_0^n}\int\mathrm{f}(\theta)\mathrm{P}(\theta t)d\theta$$

$$\mathrm{F}_n'(t)=\int h_{n-1}(t\theta)\mathrm{f}(\theta)d\theta-\frac{1}{\lambda_0^n}\int\mathrm{P}(t\theta)\mathrm{f}(\theta)d\theta$$

If now we form double and simple strata with moment and density given by (22) we find exactly the series (21) over again.

§5.—*Formulæ of Reciprocity.*—The Green's function G(pq) admits certain theorems of reciprocity. The argument used to establish these for the ordinary potential[1] is equally valid for the generalised, the symbols having their altered significance. These relations may be stated

i. If the points p and q are both in the same region or both on the boundary

$$(23) \qquad \mathrm{G}(pq)=\mathrm{G}(qp)$$

1. Plemelj. Loc. cit., S. 395-398.

ii. If p is a point of the inner region, q of the outer, and t a point on the boundary.

$$(24)\qquad \begin{aligned}(1+\lambda)G(pq)&=(1-\lambda)G(qp)\\ G(tp)&=(1+\lambda)G(pt)\\ G(tq)&=(1-\lambda)G(qt)\end{aligned}$$

From (23) and (6) we deduce immediately that if p and q are both in the same region, or both on the boundary,

$$(25)\qquad \begin{cases}Q(pq)=Q(qp)\\ G(pq)=G(qp)\end{cases}$$

If, however, p and q are in the inner and outer regions respectively, we find on substituting from (6) in the first of (24), multiplying by $\lambda_0-\lambda$ and putting $\lambda=\lambda_0$

$$(26)\qquad \begin{cases}(1+\lambda_0)Q(pq)=(1-\lambda_0)Q(qp)\\ (1+\lambda)G(pq)=(1-\lambda)G(qp)+\dfrac{2\lambda_0}{1+\lambda_0}\cdot Q(qp)\end{cases}$$

Similarly from the second and third of (24) we find

$$(27)\qquad \begin{cases}Q(tp)=(1+\lambda_0)Q(pt)\\ Q(tq)=(1-\lambda_0)Q(qt)\end{cases}$$

and thence

$$(28)\qquad \begin{cases}G(tp)=(1+\lambda)G(pt)-\lambda_0 Q(pt)\\ G(tq)=(1-\lambda)G(qt)+\lambda_0 Q(qt)\end{cases}$$

II.—*The ordinary potential.*

§6.—*Integral Relations.*—The preceding properties are common to ordinary and generalised potentials. We know, however, that while the values $\lambda=\pm 1$, which correspond to the problems for the inner and outer regions separately, may both be characteristic numbers for the ordinary potential, they are not[1] singular for the generalised. The properties arising from the existence of these poles are then peculiar to the ordinary potential. Other special relations arise from the fact that for this potential the function $h(tp)$ satisfies the integral relation[2]

$$(29)\qquad \int h(tp)dt=2,\ 1,\ \text{or}\ 0$$

according as p is within the closed surface, on the boundary or outside, and the integration is extended over the boundary. We shall find further on a corresponding formula for the generalised potential from which this may be deduced by putting $k=0$.

Let us suppose that the boundary consists of m independent surfaces each possessing at every point a definite tangent plane and two definite principal radii of curvature. The value $\lambda=1$

1. Weatherburn. Loc. cit., § 3.

2. Plemelj. Loc. cit., S. 341-4 Another proof is by Green's Theorem as in § 9 of this paper.

is always singular. We shall assume that the surfaces are all exterior to one another, so that $\lambda=-1$ is not a characteristic number. The functions $P(ts)$ and $Q(ts)$ assume simple values at the pole $\lambda_0=1$. For the functions $\psi_1(s)$, $\psi_2(s)$, . . . , $\psi_m(s)$ are such that $\psi_r(s)$ is equal[1] to $+1$ over the rth surface and zero over all the other surfaces; while $\phi_r(t)$ is a distribution of electricity over the surfaces giving constant values over each of the surfaces and throughout each of the m inner regions. This distribution $\phi_r(t)$ has a total charge $+1$ over the rth surface, and zero over each of the others. It therefore represents the electric distribution over the m surfaces regarded as conductors, due to unit charge on the rth surface. Hence, if we use an index to denote the particular value of the pole λ_0.

$$P^{+1}(ts)=\phi_r(t) \qquad r=1,\ 2,\ .\ .\ .\ .\ m$$

according as s is on the 1st, 2nd, mth surface. Further, the function $\psi_r(p)$, being equal to the potential of a double stratum of unit moment over the rth surface, is given by

$$(30) \qquad \psi_r(p)=\int_r h(tp)dt=2,\ 1,\ \text{or}\ 0$$

according as p is within the rth surface, on its boundary, or outside that surface. The potential $\Phi_r(q)$ due to the distribution $\phi_r(q)$ is the conductor potential referred to. We shall denote it by $\Gamma_r(q)$. So that

$$(31) \qquad \begin{cases} P^{+1}(tp)=2\phi_r(t),\ \phi_r(t),\ \text{or}\ 0 \\ Q^{+1}(qp)=2\Gamma_r(q),\ \Gamma_r(q),\ \text{or}\ 0 \end{cases}$$

according as p is within the rth surface, on its boundary, or in the outer region. This of course is a particular case of (13) and (14).

We may prove several interesting properties of the functions involved in (5), (7) and (8), making use of the relation (29). If in the first of (7) we replace p by a boundary point s, multiply by dt and integrate over the boundary we find

$$\int P(ts)dt=\lambda_0\int P(\theta s)d\theta$$

Hence

$$(32) \qquad \int P(ts)dt=0 \qquad \lambda_0\neq 1.$$

By the same process we deduce from (5) that*

$$(33) \qquad (1-\lambda)\int H(ts)dt=1$$

Substituting from (6) and putting $\lambda_0=1$ we have

$$(33') \qquad (1-\lambda)\int H^{+1}(ts)dt+\int P^{+1}(ts)dt=1$$

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1. Plemelj. Loc. cit., Kap. 16.

* In (32) s may be replaced by a point p. The same may be done in (33) and (34′) provided the second member be changed to 2 for p in the inner region, and to 0 for p in the outer region. Cf. § 10.

This is an identity in λ, and $P^{+1}(ts)$ does not contain λ. We may therefore put $\lambda=1$ giving

$$\int P^{+1}(ts)dt=1$$

which may also be deduced from (31) in virtue of the properties of the distribution $\phi_r(t)$. This last relation combined with (33′) shows that

$$\int H^{+1}(ts)dt=0 \tag{34}$$

while from (32) and (33) it follows that

$$(1-\lambda)\int H(ts)dt=1 \qquad \lambda_0 \pm 1. \tag{34'}$$

This may also be proved from the first of (8), multiplying by dt and integrating over the boundary.

§7.—*Expansions.*—The second member of the equation (19a) assumes, when $\lambda_0=1$, the form

$$E(t)=f(t)-\int f(\theta)\phi_r(\theta)d\theta=f(t)-C_r$$
$$r=1, 2, \ldots, m$$

according as t is on the 1st, 2nd . . . mth surface.

The series (22a) now becomes, by (29)

$$\nu(t)=[f(t)-C_r]+\lambda[f_1(t)-C_r]+\lambda^2[f_2(t)-C_r]+\ldots \tag{35}$$

and since $\nu(t)$ now possesses no pole at $\lambda=+1$, while $\lambda=-1$ is not a singular value, this series is convergent for $|\lambda|=1$. The terms therefore decrease indefinitely, and we have for the constant C_r the value[1]

$$C_r=\underset{n=\infty}{Lt}\ f_n(t)$$
$$=\underset{n=\infty}{Lt}\int f(\theta)h_n(\theta t)d\theta$$

where t is on the rth surface. The constant C_r assumes m different constant values, one on each of the surfaces.

In (35) we may put $\lambda=\pm1$ and thus obtain the moments of the strata, which satisfy respectively the boundary problems.

$$W(t^-)=-[f(t)-C_r]$$
$$W(t^+)=f(t)-C_r$$

The singular value $\lambda=1$ also corresponds to the second problem for the inner region. The second member of (19b) for this pole takes the form

$$F(t)=f(t)-\int P(t\theta)f(\theta)d\theta$$
$$=f(t)-\phi_r(t)\int f(\theta)d\theta=f(t)$$

provided the usual condition for the inner region, viz.,

$$\int f(\theta)d\theta=0$$

be satisfied. The function $\mu(t)$ represented by (22b) now becomes

$$\mu(t)=f(t)+\lambda f_1'(t)+\lambda^2 f_2'(t)+\ldots \tag{36}$$

[1] Cf. Plemelj. Potentialtheoretische Untersuchungen, S. 60.

It has no pole at $\lambda=+1$, while $\lambda=-1$ is not a singular value. The series is therefore convergent for $|\lambda|=1$. In (36) we may put $\lambda=\pm 1$ and thus obtain the densities of the simple strata which satisfy respectively the boundary problems

$$\frac{dV}{dn}(t^+)=-f(t)$$

$$\frac{dV}{dn}(t^-)=f(t).$$

The series for the solutions (21) may be obtained from that equation by substituting the values of $P(\theta p)$ and $Q(p\theta)$. Further, if $\lambda_0=1$, the functions $H^{+1}(ts)$ and $G^{+1}(ps)$ given by (20) have no pole at $\lambda=1$, while $\lambda=-1$ is not a singular value. The series are therefore convergent for $|\lambda|=1$, so that the terms decrease indefinitely. It follows that

$$P^{+1}(ts)=\phi_r(t)=\underset{n=\infty}{\mathrm{Lt}}\ h_n(ts)$$

giving the electric distribution[1], $\phi_r(t)$ in terms of the iterated functions $h_n(ts)$: the limit assuming one of m different values, according to the surface upon which s lies. Similarly from the convergence of the second series (20) for $|\lambda|=1$, it follows that

$$Q^{+1}(ts)=\underset{n=\infty}{\mathrm{Lt}}\ g_n(ts)$$

i.e.

$$\Gamma_r(t)=\underset{n=\infty}{\mathrm{Lt}}\ g_n(ts) \tag{37}$$

giving the conductor potential $\Gamma_r(t)$ as the limit of the sequence $g_1(ts)$, $g_2(ts)$, . . . which assumes m different values according to the surface on which s lies.

§8.—*Solution of the second boundary problem for both inner and outer regions in terms of a single function.*—In the second boundary problem the values $\lambda=\pm 1$ correspond to the inner and outer regions respectively. The former of these values is the only pole involved. The boundary problem (19b) becomes, for $\lambda_0=1$, and $\lambda=\pm 1$ equivalent to the separate problems represented by

$$\begin{cases} \dfrac{dV}{dn}(t^+)=-f(t) & \text{for } \lambda=+1 \\ \dfrac{dV}{dn}(t^-)=f(t) & \text{for } \lambda=-1 \end{cases} \tag{38}$$

where in the former the boundary function $f(t)$ is subject to the usual integral condition. The solutions to the problems given by (18) may be written

$$V(p)=\int G^{+1}_{+1}(p\theta)f(\theta)d\theta \tag{39}$$

and

$$V(p)=\int G^{+1}_{-1}(p\theta)f(\theta)d\theta$$

1 Cf. Potentialtheoretische Untersuchungen S. 59.

respectively, where the index represents the pole $\lambda_0 = +1$ and the suffix the particular value of λ. As now the pole $\lambda_0 = +1$ is the only one to be considered we may drop the index in what follows. These two solutions are expressed in terms of different functions $G_{+1}(ps)$ and $G_{-1}(ps)$. It is our object to express both of these in terms of a single function. By means of the second equation (8) we may write

$$(40) \qquad \begin{cases} G_{+1}(ps) - \int G_{+1}(p\theta)h(\theta s)d\theta = g(ps) - \Gamma(p) \\ G_{-1}(ps) + \int G_{-1}(p\theta)h(\theta s)d\theta = g(ps) - \Gamma(p) \end{cases}$$

If we put

$$\begin{cases} 2R(ps) = G_{+1}(ps) + G_{-1}(ps) \\ 2R_1(ps) = G_{+1}(ps) - G_{-1}(ps) \end{cases}$$

we obtain from the preceding by adding and subtracting

$$(41) \qquad \begin{cases} R(ps) - \int R_1(p\theta)h(\theta s)d\theta = g(ps) - \Gamma(p) \\ R_1(ps) - \int R(p\theta)h(\theta s)d\theta = 0 \end{cases}$$

This last equation expresses $R_1(ps)$ in terms of $R(ps)$; hence we may determine both $G_{+1}(ps)$ and $G_{-1}(ps)$ in terms of the single function $R(ps)$. From (41) we find that $R(ps)$ satisfies the integral equation

$$R(ps) - \int R(p\theta)h_1(\theta s)d\theta = g(ps) - \Gamma(p).$$

As in §4, by the method of successive approximations, this integral equation gives us an expansion for $R(ps)$ and hence for $R_1(ps)$. We find

$$\begin{cases} R(ps) = [g(ps) - \Gamma(p)] + [g_2(ps) - \Gamma(p)] + [g_4(ps) - \Gamma(p)] + \ldots \\ R_1(ps) = [g_1(ps) - \Gamma(p)] + [g_3(ps) - \Gamma(p)] + \ldots \end{cases}$$

which are both convergent, being identical with those obtained by adding and subtracting the absolutely convergent series for $G_{+1}(ps)$ and $G_{-1}(ps)$.

The solutions of the second boundary problem for both the inner and the outer regions could also be expressed in terms of the function $K(ts)$ introduced by Plemelj.[1] For from (8) we find

$$G^{+1}(ps) - \lambda \int g(p\theta)H^{+1}(\theta s)d\theta = g(ps) - \Gamma(p)$$

In this we may put $\lambda = \pm 1$ in turn, and thus obtain $G_{+1}(ps)$ and $G_{-1}(ps)$ in terms of $H_{+1}(ts)$ and $H_{-1}(ts)$ respectively, and hence in terms of $K(ts)$. Introducing the values of the functions we find

$$\begin{aligned} G_{+1}(ps) &= g(ps) - \Gamma(p) + \int g(p\theta)\{K(\theta s) + \int h(\theta\sigma)K(\sigma s)d\sigma\}d\theta \\ &= g(ps) - \Gamma(p) + \int g(p\theta)K(\theta s)d\theta + \int g_1(p\theta)K(\theta s)d\theta \end{aligned}$$

Similarly

$$G_{-1}(ps) = g(ps) - \Gamma(p) - \int g(p\theta)K(\theta s)d\theta + \int g_1(p\theta)K(\theta s)d\theta$$

So that the solutions for both regions may be expressed in terms of $K(ts)$.

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1. Potent. Unter. S. 79.

III.—*The generalised potential.*

§9.—*Fundamental formula.*—The simple forms taken by the integrals of §6 depend upon the formula (29), which is true only for the ordinary potential. I now propose to find the value of the integral

$$\int h(tp)dt$$

when the potential is generalised, corresponding to the equation (2). In Green's formula

$$(42)\qquad \int\left(\frac{dU}{dx}\cdot\frac{dV}{dx}+\ldots\right)dq=-\int U\frac{dV}{dn}ds-\int U.\nabla^2V.dq$$

put $U=1$, and $V=g(qp)$, q being a variable point and p a fixed point. If in (42) the integration is extended over a closed surface and p is outside the surface we find, since $g(qp)$ satisfies (2)

$$(43)\qquad \int h(tp)dt=-k^2\int g(qp)dq$$

where dq is the element of volume at q. The integration in the second member being extended throughout the volume enclosed by the surface, the integral represents the potential at p due to a uniform distribution of mass of unit density throughout that volume. We shall denote this potential by $X(p)$.

If, however, p is inside the closed surface we must surround p by a small sphere Ω of radius r, the surface integration of (42) now including the surface of this sphere, and the volume integration extending only throughout the volume between the sphere and the original surface. At the small sphere the positive direction of the normal is that of r increasing, so that (42) becomes

$$\int h(tp)dt+k^2\int(gp)dq=-\int_\Omega\frac{d}{dr}g(sp)ds=1/2\pi\int_\Omega e^{-kr}\left(\frac{k}{r}+\frac{1}{r^2}\right)ds$$

and when the radius of the sphere becomes vanishingly small the second member is equal to 2. Hence when p is within the closed surface

$$(44)\qquad \int h(tp)dt=2-k^2\int g(qp)dq=2-k^2X(p)$$

the volume integral of the second member being convergent[1] since the subject of integration becomes infinite at $p=q$ only as $1/r$.

To find the value of $\int h(ts)dt$ where s is a point on the boundary we observe that $\int h(tp)dt$ is a double stratum potential of unit moment over the boundary. Hence its value at a point on the surface is the mean of its values at points infinitesimally close to this, one just inside and the other just outside. So that

$$(45)\qquad \int h(ts)dt=1-k^2\int g(qs)dq$$
$$=1-k^2X(s)$$

1 Cf. Leathem. "Volume and surface integrals used in Physics," p. 14 (Cambridge Tract, 1905).

§10.—*Further relations.*—By means of the preceding results we may obtain relations corresponding to those of §6 for the ordinary potential. From the first equation (7) we find on multiplying by dt and integrating over the boundary,

$$\int P(tp)dt = \lambda_0 \int [1-k^2X(\theta)]P(\theta p)d\theta$$

that is,

$$(46)\qquad (1-\lambda_0)\int P(tp)dt = -k^2\lambda_0 \int X(\theta)P(\theta p)d\theta$$

which reduces to (32) when k^2 is put equal to zero. Similarly from the first of (8) we find on integration with respect to t

$$\lambda \int [1-k^2X(\theta)]H(\theta p)d\theta = \int H(tp)dt - c + k^2X(p) - \frac{1}{\lambda_0}\int P(tp)dt$$

or

$$(47)\qquad (1-\lambda)\int H(tp)dt = c - k^2X(p) - \lambda k^2 \int X(\theta)H(\theta p)d\theta + \frac{k^2}{1-\lambda_0}\int X(\theta)P(\theta p)d\theta$$

where c has the value 2, 1, or 0, according as p is within the inner region, on the boundary, or in the outer region. This relation reduces to (33) when k is zero and p on the boundary.

These might have been derived from (5), the first of which becomes on integration

$$(48)\qquad (1-\lambda)\int H(tp)dt = c - k^2X(p) - \lambda k^2 \int X(\theta)H(\theta p)d\theta$$

Substituting from (6), multiplying by $(\lambda_0-\lambda)$ and proceeding to the limit $\lambda=\lambda_0$ we arrive at (46). Then substituting from this in (48) we find (47).

The preceding investigation deals with the singular parameter values of the first two boundary problems only. In another paper[1] the author considers the third boundary problem for the equation (2), requiring the determination of a solution satisfying the relation

$$\frac{dV}{dn}(t^+) = \lambda\beta(t)V(t^+) - \beta(t)U(t)$$

The singular parameter values for this problem are there discussed.

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1 Weatherburn. "The mixed boundary problem for the generalised potential corresponding to the equation $\nabla^2 u - k^2u=0$." Quarterly Journal, vol. 46, pp. 83-94.